On OM Algorithms and Cluster Pictures

Adrien Poteaux
Université Lille; Cristal
Lille, France
adrien.poteaux@univ-lille.fr

Tristan Vaccon
Université de Limoges; CNRS, XLIM
UMR 7252
Limoges, France
tristan.vaccon@unilim.fr

Martin Weimann
Université de Caen-Normandie ;
LMNO
Caen, France
martin.weimann@unicaen.fr

ABSTRACT

In this paper, we study the connection between the OM-factorization of a polynomial and its cluster pictures, which is a representation of the relative configuration of the roots. Our contribution is threefold, assuming that the residual characteristic zero or large enough: (1) We provide and showcase an implementation of the OM algorithms. (2) We make explicit and constructive the connection between the valuative tree of a polynomial, the cluster picture of its roots and the Berkovich skeleton of its roots. As such, we provide a complexity result on the computation of cluster pictures. (3) We elaborate on this connection to provide and showcase an algorithm to compute cluster pictures based on the OM algorithms.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Algorithms, OM algorithms, augmented valuation, inductive valuation, cluster picture, Berkovich skeleton

ACM Reference Format:
Adrien Poteaux, Tristan Vaccon, and Martin Weimann. 2025. On OM Algorithms and Cluster Pictures. In *International Symposium on Symbolic and Algebraic Computation (ISSAC '25), July 28 – August 1, 2025, Guanajuato, Mexico.* ACM, New York, NY, USA, 10 pages. https://doi.org/XXXXXXXX

1 INTRODUCTION

Let (K, v_K) be a discrete rank-one valued field with henselization K^h. Given a separable irreducible polynomial $g \in K[x]$, the OM algorithm computes an approximation of each irreducible factors of g in $K^h[x]$. These factors are one-to-one with the extensions of v_K to the field $K[x]/(g)$, and a crucial feature of the OM algorithm is to approximate these valuations by some inductive valuations on $K[x]$, which are some suitably increasing sequences of extended valuations on $K[x]$.

It turns out that there is an equivalent description of the involved extended valuations on $K[x]$ in the language of rigid analytic geometry, valuations corresponding to rigid diskoids [29, Thm 4.56]. Besides establishing a bridge between valuation theory and non archimedean geometry, this reinterpretation gives a convenient way to recover the *cluster pictures* of a polynomial $g \in K[x]$ from its OM-factorization when the ramification is tame. This is the main topic of this paper. Cluster pictures have been introduced in [7] as a convenient representation of the relative configuration set of the roots of a polynomial to compute various arithmetic data attached to hyperelliptic curves: semitable model, conductor, minimal discriminant, Galois representation, Tamagawa number, root number, reduction type, Kodaira type, ... We refer to [6, 13] for an extensive presentation of the applications of cluster pictures (see also [13]). This reference has been followed by a SageMath package[1]. Independently, a theoretical approach has been presented in [12]. However, no theoretical analysis nor complexity of the computation of cluster pictures has been presented in these previous works.

1.1 Contributions.

In what follows, we let A be the valuation ring of a discrete rank-one valued field (K, v_K). We assume that A is principal. We denote by κ the residue field of K.

In this work, we show that computing the cluster picture of a polynomial $g \in K[x]$ mainly reduces to compute an OM-factorization under the assumption that no wild ramification occurs. More precisely, this paper is dedicated to three main tasks, assuming that the residual characteristic is zero or large enough :

(1) We provide and showcase an implementation of the OM algorithm.
(2) We make explicit and constructive the connection between the OM factorisation of g, the valuative tree (Section 5.1), the cluster picture of its roots (Section 3) and the Berkovich skeleton (Section 6).
(3) We elaborate on this connection to provide and showcase an algorithm to compute cluster pictures based on the OM algorithms.

Combined with [28], this leads to the following complexity estimates for cluster pictures and Berkovich skeleton :

Theorem 1.1. *There exists a deterministic algorithm which, given $g \in A[x]$ monic separable of degree d such that* $\mathrm{char}(\kappa) = 0$ *or* $\mathrm{char}(\kappa) > d$*, computes the cluster picture and the Berkovich skeleton of the roots of g with $O_\varepsilon(d\delta)$ operations in κ where $\delta = \delta(g)$ is the generalized Okutsu bound[2], plus some residual univariate factorizations whose costs fit in the bound if $\kappa = \mathbb{F}_p$.*

We use here the notation $O_\varepsilon(d) := O(d^{1+o(1)})$ where $O()$ denote the classical asymptotic notation. We work with computation trees and we use an algebraic RAM model, counting only the number of arithmetic operations in κ.

[1] https://github.com/alexjbest/cluster-pictures/tree/master
[2] See Eq. (1) on page 5.

ISSAC '25, July 28 – August 1, 2025, Guanajuato, Mexico

ACM ISBN XXXXXXX...$15.00
https://doi.org/XXXXXXXX

Remark 1.2. *This result follows from the fact that we can easily deduce the cluster picture from an OM-factorization with suitable precision. If the residual characteristic is small, we may still compute an OM-factorization of g (with a higher complexity due to the lack of approximate roots, see Remark 4.1 and [3, 28]). However, the output is not sufficient to recover cluster pictures when wild ramification occurs (see Example 4.3).*

Remark 1.3. *By [27], the bound $\delta(g)$ is at most twice the v_K-index of the integral closure of A in $K[x]/(g)$. In particular, we have $\delta(g) \leq v_K(\mathrm{Disc}(g))$, the difference being possibly significant.*

Remark 1.4. *If $g \in A[x]$ is not monic, a single Hensel lifting allows to reduce to the monic case. However, we need then to take care of the valuation of the leading coefficient of g in the bound $\delta(g)$ (see e.g. [26] for such considerations in the context of Puiseux series).*

1.2 History and related results

History. The OM algorithm is inspired from a pioneering work of Ore in the 1920s [24, 23] about the factorization of a prime number p in a number field $\mathbb{Q}[x]/(g)$. Ore conjectured the existence of an algorithm based on the iteration of a double dissection process, in the spirit of Hensel's work : a partial factorization according to the slopes of the Newton polygon attached to some extended valuation of $\mathbb{Q}[x]$, and a further partial factorization according to the prime factors of a certain residual polynomial of g attached to each slope. In the 1930s, Mac Lane solved this problem in the more general context of a discrete rank-one field (K, v_k), introducing the central notion of inductive valuations and key polynomials [16, 15]. In the 1980s, Okutsu [21] constructed similar approximations without using valuations on $K[x]$ nor key polynomials, motivated by the computation of integral bases in local fields. In 1999, Montes [25] constructed a concrete residual polynomial operator leading to the design of a practical algorithm following the exact pattern that Ore had foreseen. This algorithm is known as the OM algorithm, named after Ore, Mac Lane, Okutsu and Montes.

Complexity. Up to our knowledge, the best complexity results for the OM algorithm is given in [28], thanks to a fast Hensel lifting with respect to an extended valuation, and a divide and conquer strategy based on a suitable care of the precision. Moreover, when the residual characteristic is zero or high enough (as assumed in this paper), Abhyankar's approximate roots provide some easy-to-compute *optimal* key polynomials, allowing to decrease the number of iterations of the double dissection process. This leads to the complexity stated in Theorem 1.1.

Generalization to non discrete rank-one valuations. Besides complexity issues, approximate roots are crucial objects since they allow to generalize the OM algorithm for some non discrete rank-one valued fields [1], based on the modern theory of Mac Lane-Vaquié valuations, [32, 18]. Let us mention that there does not exist nowadays an OM algorithm for non discrete or higher rank valuations in arbitrary residual characteristic. This open problem is of particular importance with regards to its deep interplay with the resolution of singularities in positive characteristic, see e.g. [20, 19].

Algorithmic of global fields. Besides factoring polynomials over Henselian fields, the OM algorithm leads to very efficient resolution of many arithmetic-geometric tasks in number fields and function fields of algebraic curves, such as factoring ideals in Dedekind rings (which was the original motivation of Ore) or computing integral basis [9, 11, 27]. A package '+Ideals' for Magma has been designed for this purpose in the case of number fields [10].

Arithmetic geometry and clusters. Mac Lane valuations and key polynomials play also a key role in arithmetic geometry since we can use valuations to represent a normal model of a curve X over a valued field K, a key point towards the computation of a semistable model of X (that is an integral, proper, and flat scheme over the valuation ring of K with generic fiber X and whose special fiber is reduced with ordinary double points as singularities). This fact is illustrated in the case of hyperelliptic curves in [13, 29]. It turns out that many computational tasks of arithmetic geometry (semistable model, lattice of regular forms, regulator, etc.) are facilitated once we know the cluster picture of the input polynomial (see e.g. [13, 6, 7]), this combinatorial object being a convenient representation of the relative v_K-adic distances between the roots. In turn, the connection between valuations and cluster pictures requires the language of rigid analytic geometry. On one hand, there exists a bijection between augmented valuations on $K[x]$ with residual transcendental extensions and rigid diskoids with finite radius [29, Thm 4.56], and on the other hand it is shown in [13] that we can recover the cluster pictures from these diskoids (at least in the tamely ramified case). The possibility of using various languages for somehow a same object is of theoretical and practical interest, as it relates valuation theory, arithmetic geometry and Berkovich geometry. As such, we believe that it is an important issue to design efficient algorithms to switch from one representation (valuations, clusters, or skeleton) to another.

1.3 Organisation.

In Section 2, we define the inductive valuations of $K[x]$ as introduced by Mac Lane [16, 15], and we explain how to represent them with diskoids, following [29]. We define the cluster picture of a polynomial $g \in K[x]$ in Section 3, providing some illustrative examples. In Section 4, we detail the OM algorithm of [28] (assuming residual characteristic zero or large enough), and we give some first illustrations of its connection with clusters. Example 4.3 shows that assuming tame ramification is unavoidable in this context. Section 5 constitutes the main part of this paper : we explain how to compute the cluster picture of a polynomial g from an OM-factorization with a suitable precision. To this aim, we use the notion of valuative tree and we relate on Proposition 5.9 to compute the relative depths of the clusters. We obtain in such a way a complexity estimates for the computation of cluster pictures (Theorem 5.11). In the last Section 6, we deduce an algorithm which computes the Berkovich skeleton of a polynomial g, which is a certain sub-tree of the Berkovich unit disk representing the roots of g, giving a new representation of an OM-factorization in the language of rigid analytic geometry.

The algorithms presented in this document have been implemented in SageMath [31] and are available at https://gist.github.com/TristanVaccon. The bibliography is followed by an Annex Section where we showcase the principal characteristics of our implementations.

2 AUGMENTED VALUATIONS AND DISKOIDS

2.1 Inductive valuations

Let $V(K[x])$ denote the set of discrete valuations on $K[x]$ which extend v_K and satisfy $v(x) \geq 0$. The set $V(K[x])$ can be equipped with the partial order: $v \leq v'$ if for any $f \in K[x]$, $v(f) \leq v'(f)$.

Example 2.1. *The* Gauss valuation, v_0 *defined by*

$$v_0\left(\sum_i a_i x^i\right) := \min_i \{v_K(a_i)\},$$

is a minimal element of $V(K[x])$ *with respect to the partial order* $\leq$.

Definition 2.2. *Let* $v \in V(K[x])$ *and* $f, g, h \in K[x]$.

(1) f, g *are* v*-equivalent* ($f \sim_v g$) *if* $v(f-g) > v(f) = v(g)$.
(2) f v-divides g ($f|_v\, g$) *if* $g \sim_v ff'$ *for some* $f' \in K[x]$. *It is* v-irreducible *if* $f|_v\, gh$ *implies* $f|_v\, g$ *or* $f|_v\, g$ *and* v-minimal *if* $f|_v\, g$ *implies* $\deg(f) \leq \deg(g)$.
(3) $\phi \in K[x]$ *is called a* key polynomial *for* v *if* ϕ *is monic, integral,* v*-irreducible and* v*-minimal.*

Definition 2.3. *Let* $v \in V(K[x])$, ϕ *a key polynomial over* v *and* $\lambda > v(\phi)$. *We define the* augmented valuation $v' = [v, v'(\phi) = \lambda] \in V(K[x])$ *of* v *with respect to* (ϕ, λ) *as follows. Let* $f \in K[x]$ *and write its unique* ϕ*-expansion as* $f = \sum_i f_i \phi^i$ *(that is* $f_i \in K[x]$ *and* $\deg(f_i) < \deg(\phi)$ *for all* i*). Then* $v'(f) := \min_i (v(f_i) + i\lambda)$.

Definition 2.4. *Starting from the Gauss valuation* v_0*, and given polynomials* $\phi_1, \ldots, \phi_n$ *with* $\deg(\phi_1) < \cdots < \deg(\phi_n)$ *and positive rationals* $\lambda_1 < \cdots < \lambda_n$*, we define recursively the* inductive valuation

$$v_n = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_n(\phi_n) = \lambda_n]$$

via $v_k := [v_{k-1}, v_k(\phi_k) = \lambda_k]$ *for* $k = 1, \cdots, n$*, as long as* ϕ_k *is a key polynomial for* v_{k-1}.

Approximating valuations of L/K *by inductive valuations.* We fix an algebraic closure $\overline{K}$ of K and an extension of v_K to $\overline{K}$. This determines an henselization $K \subset K^h \subset \overline{K}$, and we abusively still denote v_K the unique extension of v_K to these valued fields. Any irreducible polynomial $f \in K^h[x]$ induces a quasi-valuation w_f on $K[x]$ by $w_f(q) := v_K(q(\alpha))$ for α an arbitrary root of f. The quasi-valuation w_f has kernel $gK[x]$ where g is the minimal polynomial of α over K, hence induces a valuation $\overline{w}_f$ on the field extension $L = K[x]/(g)$. This construction establishes a one-to-one correspondence between the irreducible factors f of g in $K^h[x]$ and the extensions of v_K to the field L. We refer to [1] for details. Given $g \in K[x]$, the OM-algorithm computes for each factor f an inductive valuation which approximates w_f from which we can deduce the ramification index and the residual degree of $\overline{w}_f$.

2.2 Data attached to inductive valuations

Definition 2.5. *Let* $v \in V(K[x])$ *be a valuation. The* valuation ring *of* v *is* $O_v := \{f \in K[x] : v(f) \geq 0\}$. *Its prime ideal is* $O_v^+ := \{f \in K[x] : v(f) > 0\}$ *and its* residue ring $\mathfrak{R}_v := O_v/O_v^+$.

The valuation v naturally extends to the field $K(x)$ and the residue field of $(K(x), v)$ is the field of fraction of $\mathfrak{R}_v$.

Definition 2.6. *Let* $v_n = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_n(\phi_n) = \lambda_n]$ *be an inductive valuation.*

(1) *The value group of* v_n *is the subgroup* $\Gamma_n \subset \mathbb{Q}$ *generated by* $v_n(K[x])$. *The* relative ramification index *of* v_n *is* $e_n := (\Gamma_n : \Gamma_{n-1}) \in \mathbb{N}$.
(2) *For all* i, $\mathfrak{R}_{v_i}$ *is isomorphic to a polynomial ring* $\kappa_{v_i}[y_i]$ *for some finite field extension* κ_{v_i} *of* κ*, with* $\kappa_{v_0} = \kappa$. *The field* κ_{v_i} *is the relative algebraic closure of* κ *in the residue field of* v_i.
(3) *For all* $i \geq 1$, *the field* κ_{v_i} *is a finite extension of* $\kappa_{v_{i-1}}$ *and we define the* relative residual degree *of* v_i *as* $f_i := [\kappa_{v_i} : \kappa_{v_{i-1}}]$.

2.3 diskoids

It is well known that the norm function counterpart to the Gauss valuation v_0 is the maximum norm on the unit disk. There is a generalization to this result to any inductive valuation, involving diskoids, a generalization of discs. The connection between inductive valuations and diskoids has been pioneered by Julian Rüth in his seminal PhD-thesis work [29]. These results have been extended and made numerically more precise in [5, 13, 14].

Definition 2.7. *Let* $\alpha \in K$ *and let* $\lambda \in \mathbb{Q}$. *Then* $D_K(\alpha; \lambda) := \{x \in K : v_K(x - \alpha) \geq \lambda\}$ *is the* (closed) disk *with center* α *and radius* λ *over* K. *The index* K *will be omitted when the context is clear.*

Definition 2.8. *Let* $\phi \in K[x]$ *be a monic irreducible polynomial and let* $\lambda \in \mathbb{Q}$. *Then* $D_K(\phi; \lambda) := \{x \in K : v_K(\phi(x)) \geq \lambda\}$ *is the* diskoid *with center* ϕ *and radius* λ *over* K. *The index* K *will be omitted when the context is clear.*

When extending scalars to $\overline{K}$, diskoids become union of disks:

Lemma 2.9 ([29, Lem. 4.43]). *Let* $\phi \in K[x]$ *be a monic irreducible polynomial, let* $\lambda \in \mathbb{Q}$ *and let* $\alpha_1, \ldots, \alpha_d$ *be the roots of* ϕ *over* $\overline{K}$. *Then there exists some* $\gamma \in \mathbb{Q}$ *such that:*

$$D_{\overline{K}}(\phi, \lambda) = \cup_{i=1}^d D_{\overline{K}}(\alpha_i, \gamma).$$

Thanks to [29], there is an explicit bijection between inductive valuations and diskoids. The main point is the following.

Theorem 2.10 ([29, Th. 4.56]). *Let* $v_n = [v_0, \ldots, v_n(\phi_n) = \lambda_n]$ *be an inductive valuation. Then*

$$\forall f \in K[x],\ v_n(f) = \inf \{v_K(f(x)),\ \text{for } x \in D_K(\phi_n, \lambda_n)\}.$$

This result is the extension of the fact that: $\forall f \in K[x]$, $v_0(f) = \min \{v_K(f(x)),\ \text{for } x \in D_K(0, 0)\}$.

3 CLUSTER PICTURES

We follow the definitions of [13, §3.2]. Let $f = c_f \prod_{i=1}^d (c - \alpha_i) \in K[x]$ be a polynomial with $c_f \in K$ and $\alpha_1, \ldots, \alpha_d \in \overline{K}$. We denote by $\mathfrak{R}$ the set of roots of f: $\mathfrak{R} = \{\alpha_1, \ldots, \alpha_d\}$.

Definition 3.1. (1) *A* cluster *is a non-empty subset* $\mathfrak{s} \subset \mathfrak{R}$ *of the form* $\mathfrak{s} = D \cap \mathfrak{R}$ *for some disk* $D = D_{\overline{K}}(\alpha, \lambda)$ *with* $\alpha \in \overline{K}$ *and* $\lambda \in \mathbb{Q}$. *We call* α *a* center *of the cluster.*
(2) *If* $|\mathfrak{s}| > 1$ *then* $\mathfrak{s}$ *is called a* proper *cluster and its depth is defined to be* $d_\mathfrak{s} := \min_{\alpha, \alpha' \in \mathfrak{s}} v_K(\alpha - \alpha')$.
(3) *If* $\mathfrak{s} \neq \mathfrak{R}$, *we let* $P(\mathfrak{s})$ *be the smallest cluster strictly containing* $\mathfrak{s}$ *and call it the* parent *of* $\mathfrak{s}$. *Conversely,* $\mathfrak{s}$ *is a* child *of* $P(\mathfrak{s})$.
(4) *The* relative depth *of a proper cluster* $\mathfrak{s}$ *is defined as* $\delta_\mathfrak{s} := d_\mathfrak{s} - d_{P(\mathfrak{s})}$. *In addition, we set* $\delta_\mathfrak{R} := d_\mathfrak{R}$.

(5) *The* cluster picture *of f, denoted by Σ_f is the set of all clusters of $\mathfrak{R}$ with its partial-order set structure provided by the parent (or inclusion) relation.*

Definition 3.2. *(1) A set of clusters $\mathfrak{o} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\}$ is called a* Galois orbit of clusters *if the absolute Galois group G_K acts transitively on $\mathfrak{o}$. In this case, because the action of G_K is isometric, all the $\mathfrak{s}_i$'s have the same depth, denoted by $d_{\mathfrak{o}}$.*

(2) Let $\mathfrak{o}$, $\mathfrak{o}'$ be two Galois orbits. We say that $\mathfrak{o}$ is the parent *of $\mathfrak{o}'$ if for very every cluster $\mathfrak{s}' \subset \mathfrak{o}'$ there exists a cluster $\mathfrak{s} \subset \mathfrak{o}$ such that $\mathfrak{s}' \subset \mathfrak{s}$ and $\mathfrak{o}$ is the smallest cluster orbit which satisfies this property. We also say that $\mathfrak{o}'$ is a* child *of $\mathfrak{o}$.*

(3) We write $P^*(\mathfrak{o})$ for the smallest cluster containing all clusters in the Galois orbit $\mathfrak{o}$.

Definition 3.3. *(1) For a cluster $\mathfrak{s} \subset \mathfrak{R}$, let $D_{\mathfrak{s}} := D_{\overline{K}}(\alpha_{\mathfrak{s}}, d_{\mathfrak{s}})$, where $\alpha_{\mathfrak{s}} \in \overline{K}$ is a center of $\mathfrak{s}$ and $d_{\mathfrak{s}}$ is the depth of $\mathfrak{s}$. We say that $D_{\mathfrak{s}}$ is the* disk *associated to $\mathfrak{s}$.*

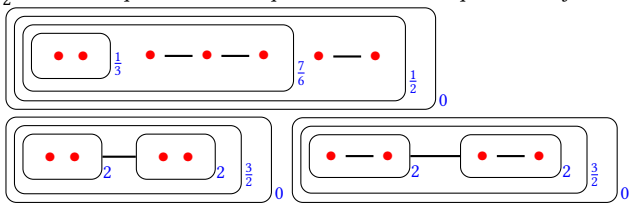
(2) Let $\mathfrak{o} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\}$ be a Galois orbit of clusters. Let g be the factor of f with set of roots equal to $\mathfrak{s}_1 \cup \dots \cup \mathfrak{s}_m$. We define

$$D_{\mathfrak{o}} := D(g, \lambda), \text{ where } \lambda = \min\left\{g(\alpha) | \alpha \in D_{\mathfrak{s}_1} \cup \dots \cup D_{\mathfrak{s}_m}\right\}$$

*and say that $D_{\mathfrak{o}}$ is the K-*diskoid *associated to the orbit $\mathfrak{o}$.*

Representation of cluster pictures. The clusters can be represented by embedded boxes with subscript indicating the relative depth, except for the top box for which it is its depth. The roots are represented by red dots. There is a line between clusters or between roots to represent a Galois orbit: those connected clusters or connected roots are permuted by Galois action.

Example 3.4. *Let $f_{1,1} = (x^2 - t)(x - t^2)(x - 2t^2)(x^3 - t^5)$, $f_{1,2} = (x^2 - t^3)^2 - t^{10} = (x^2 - t^3 - t^5)(x^2 - t^3 + t^5)$ and $f_{1,3} = (x^2 - t^3)^2 + t^{10}$ be polynomials in $\mathbb{Q}[t][x]$. Because of the factorisation into two factors over $\mathbb{Q}[t]$ of $f_{1,2}$, there is two disjoints Galois orbits of roots and one Galois orbit of two clusters at depth $\frac{7}{2}$. In contrary, the four roots of $f_{1,3}$ are in one Galois orbit which splits into two cluster orbits at depth $\frac{7}{2}$. One can represent their respective three cluster pictures as follows.*



4 OM ALGORITHMS

We assume that the valuation ring A of K is principal, and we consider for simplicity $g \in A[x]$ a monic separable polynomial. Recall that K^h stands for an Henselization of (K, v_K). The OM-algorithm computes an approximation of the irreducible factors $g_1, \dots, g_r$ of g in $K^h[x]$. As explained at the end of Section 2.1, each g_i determines a valuation w_{g_i} on $K[x]$. The OM-algorithm will compute some inductive valuations v_i close enough to the valuations w_{g_i} to deduce the cluster picture of g.

4.1 Required Tools Already Available

Reduction. Given an inductive valuation μ, one can compute the residual ring morphism surjection: $K[x] \twoheadrightarrow \mathfrak{R}_v = O_v / O_v^+$, whose kernel is O_v^+. It can be extended into an application $R_\mu : K[x] \twoheadrightarrow \mathfrak{R}_v$ s.t. for any $g, h \in K[x]$, $R_\mu(gh) = R_\mu(g)R_\mu(h)$ and key polynomials of μ are sent to irreducible elements of $\mathfrak{R}_v$. We refer to [8, 22] and [29, §4.1.3] for more details on how to compute R_μ in practice.

Liftings. We assume conversely that there is a lifting procedure from $\mathfrak{R}_v$ to $K[x]$ sending monic irreducible polynomials to key polynomials. We also assume that there is a `Hensel` procedure taking as parameters $g, \mu, [\phi_0^{n_0}, \dots, \phi_s^{n_s}])$, σ such that $\mu(g) = \mu(\phi_0^{n_0}\phi_s^{n_s})$ and $\mu(g - \phi_0^{n_0}\phi_s^{n_s}) - \mu(g) > 0$ and providing $G_0^{(\sigma)}, \dots, G_n^{(\sigma)}$ such that for all i, $R_\mu(G_i^{(\sigma)}) = \phi_i^{n_i}$ and $\mu(g - G_0^{(\sigma)}G_s^{(\sigma)}) - \mu(g) > \sigma$. See [28, §4].

Generalized Newton Polygons. Let μ be an inductive valuation, $\phi \in K[x]$ a key polynomial for μ and $g \in K[x]$. If $g = a_n\phi^n + a_{n-1}i^{n-1} + \dots + a_1\phi + a_0$ is its ϕ-adic development, the generalized Newton polygon $N_{\mu,\phi}(g)$ is the upper convex hull of the points $(i, \mu(a_i))$. As for the classical Newton polygon, slopes of $N_{\mu,\phi}(g)$ are directly connected to the roots of g and its factorization.

4.2 Approximate roots

Since [28, 1] it is known that some of the key polynomials to consider in the OM algorithms can be efficiently produced using approximate roots. A monic polynomial $Q \in K[x]$ is an n-th *approximate root* of a monic polynomial $g \in K[x]$ if the Q-adic expansion of g (i.e. $\deg(a_i) < \deg(G)$) has no $(n-1)^{th}$ coefficient:

$$g = Q^n + a_{n-2}Q^{n-2} + \dots + a_1Q + a_0$$

Note $\deg(Q) = \frac{\deg(g)}{n}$. When $n | \deg(g)$ and $\mathrm{char}(K) \nmid \deg(g)$, the n-th approximate root exists and is unique. It can be computed very naturally as ψ, starting from $\psi = x^{\deg(g)/n}$ and repeating: (1) Write $g = \psi^n + a_{n-1}\psi^{n-1} + \dots + a_1\psi + a_0$, the ψ-expansion of g, (2) $\psi \leftarrow \psi + \frac{a_{n-1}}{n}$, until $a_{n-1} = 0$. Thanks to a Newton operator, a faster algorithm in softly-linear complexity is obtained in [26].

4.3 Irreducibility Test

Algorithm 1 builds an inductive valuation that proves if an entry polynomial is irreducible or not. It is a first step towards an OM-factorization. From [27, Thm 2], we can perform all computations using a v_K-adic precision $2\delta(g)/\deg(g)$, where $\delta(g)$ is defined in Eq. (1). This implies a complexity $O_\varepsilon(\delta(g))$ [28].

Remark 4.1. *If $\mathrm{char}(\kappa)$ divides $\deg(g)$, either approximate roots do not exist, or they are not key polynomials. In such a case, we rather compute at step 8 a (non canonical) representative of ψ, that is a monic polynomial $\phi \in A[x]$ whose residual polynomial is ψ. In contrast to approximate roots, we get a key polynomial which is not necessarily optimal : the next computed key polynomial, say ϕ', may satisfy $\deg(\phi) = \deg(\phi')$, leading to a so-called refinement step. The algorithm terminates anyway, but the complexity is $O_\varepsilon(\delta(g)^2)$ (see [3]). Unfortunately, these considerations are not sufficient to recover the cluster pictures from an OM-factorization when wild ramification occurs, see Example 4.3 below.*

Algorithm 1: OM-Irreducibility

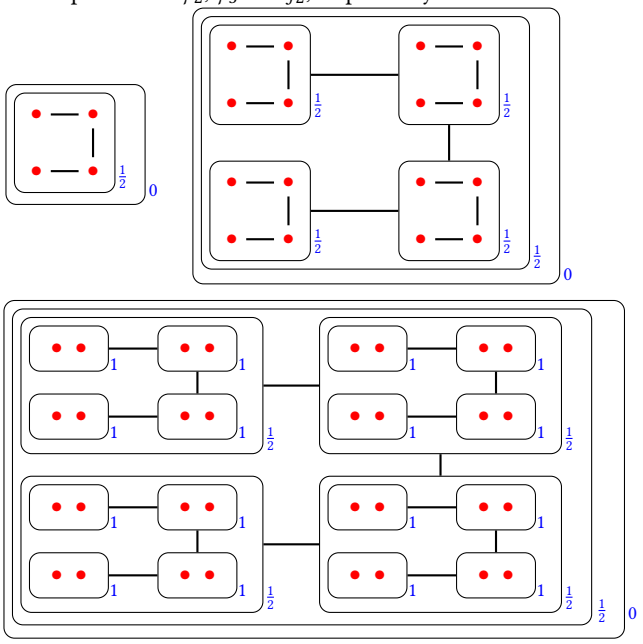
input : $g \in A[x]$ monic and separable. Assume $\mathrm{char}(\kappa) \nmid \deg(g)$ (otherwise use Remark 4.1).

output : A boolean expressing whether g is irreducible or not in K^h, an inductive valuation proving the irreducibility or that g splits, and a key polynomial

$\mu \leftarrow \mu_0, \phi \leftarrow t, n \leftarrow \deg(g)$; `// $\mu_0$: Gauss valuation`
while $n > 1$ **do**
  if $N_{\mu,\phi}(g)$ *is one-sided (of slope* $-\lambda$*)* **then**
    $\mu \leftarrow \mu' = [\mu; \mu'(\phi) = \lambda]$ `// augmented valuation`
  else return False, μ, ϕ;
  if $R_{\mu,\phi}(g) = \psi^m$ *for some* $\psi \in \mathrm{Irr}(R_\mu)$ **then**
    $\phi \leftarrow$ `ApproximateRoot`(g, m)
  else return False, μ, ϕ;
  $n \leftarrow m$;
return True, μ, ϕ

4.4 Valuation and cluster pictures along an inductive valuation

Let us present the cluster pictures of the key polynomials obtained when applying Algorithm 1 to $f_2 := ((x^4 - 2t^2)^4 - 3t^{10})^2 - 6t^{22}$. The key polynomials are $\phi_1 = x$, $\phi_2 := x^4 - 2t^2$ and $\phi_3 := x^{16} - 8t^2x^{12} + 24t^4x^8 - 32t^6x^4 - 3t^{10} + 16t^8$. We represent below the cluster pictures of ϕ_2, ϕ_3 and f_2, respectively.

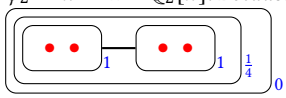


One can see that passing from ϕ_2 to ϕ_3 and ϕ_3 to f_2, roots are replaced by clusters of roots and thus, the cluster picture is refined at each step until obtaining that of f_2. This connection between cluster pictures and inductive valuations has been made precise in the following proposition.

Proposition 4.2 (Direct corollary of [13, Prop. 3.18]). *Let $v = [v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n]$ be an inductive valuation with $\deg(\phi_i)$ strictly increasing and the ϕ_i's monic and irreducible, and $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(\phi_n)$.*[3] *Then the cluster picture of ϕ_n is a chain of Galois orbits defined by the diskoids of the v_i's:*

$$D(0,0) \supset D(\phi_1, \lambda_1) \supset \cdots \supset D(\phi_{n-1}, \lambda_{n-1}).$$

Without the assumption $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(\phi_n)$, there is no 1-1 correspondence between cluster pictures and the diskoids defined by an inductive valuation.

Example 4.3 ([13, Exp. 3.19]). *Let $\phi_2 := x^4 - 2 \in \mathbb{Q}_2[x]$. Because*  *$\mathrm{val}_2(2\sqrt[4]{2}) = \frac{5}{4}$, its cluster picture is: However, ϕ_2 is irreducible and one can not get a longer chain of valuation than $[v_0, v_1(x) = \frac{1}{4}, v_2(x^4 - 2) = +\infty]$. In other words, the two clusters with two roots and relative depth 1 can not be seen on the chain of valuations.*

While Proposition 4.2 provides a connection between cluster pictures and diskoids of inductive valuations for an irreducible polynomial, it still remains to understand what happens for a non irreducible polynomial and how to obtain the data on the cluster picture. We answer those questions using Algorithm 2.

4.5 OM-factorization algorithm

Algorithm 2 provides approximation of all the irreducible factors in $K^h[x]$ and all the valuations for the valuative tree.

Precision. We need to perform computations (especially Hensel liftings) up to a suitable precision. Denote A^h the valuation ring of K^h. Let $G \in A^h[x]$ monic of degree d, separable and irreducible. Let α be a root of G. The valuation v_K extends uniquely to a valuation on the field $K^h(\alpha)$ that we still denote v_K. We define the *Okutsu bound* of G as

$$\delta_0(G) := d \max\left\{\frac{v_K(h(\alpha))}{\deg(h)},\ h \in A[x] \text{ monic, } \deg(h) < d\right\}.$$

Given $g \in A^h[x]$ monic and separable with irreducible factors $G_1, \dots, G_r \in A^h[x]$, and denoting $d_i = \deg(G_i)$, the *generalized Okutsu bound* of g is

$$\delta(g) := \frac{1}{2}\sum_i d_i \delta_0(G_i) + \sum_{i \neq j} v_K(Res(G_i, G_j)). \tag{1}$$

By [27, Thm 2], running the OM-algorithm with precision $\delta(g)$ computes for each i an approximation g_i of G_i with the complete inductive valuation leading to it, and with the extra condition $v_K(g_i(\alpha_i)) > v_k(g_i(\alpha_j))$ for all $i \neq j$, with α_i an arbitrary root of G_i. Such a data is called an *OM-factorization of g*.

The precision bound satisfies $\delta(g) \leq v_K(\mathrm{Disc}(g))$, hence improves the discriminant valuation which is traditionally considered in the literature (the gain is particularly significant when wild ramification occurs). This bound can be further improved in some particular cases (see [27, Section 2.2]), a key point for a divide-and-conquer strategy being that the bound $2\delta(g)/d$ is sufficient for irreducibility test.

[3] This last assumption is generalized in [13] into: the splitting field of ϕ_n is tamely ramified over K.

Algorithm 2: OM-Factorization

input : $g \in A[x]$ monic separable with $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(g)$, and $\sigma > \delta(g)$ a precision.

output : The irreducible factors of $g \in K^h[x]$ computed with Gauss precision $\geq \sigma$ and a set of induced valuations for which they are the leaves

1 $B, \mu, \phi \leftarrow$ OM-Irreducibility(g) ;
2 **if** B **then return** g, $\{\mu\}$;
3 $-\lambda \leftarrow$ right-end slope of $N_{\mu,\phi}(g)$;
4 $\mu_\lambda \leftarrow [\mu, \mu_\lambda(\phi) = \lambda]$;
5 Compute and factorize $R_{\mu_\lambda}(g) = \psi_1^{n_1} \dots \psi_s^{n_s} \in \mathfrak{R}_{\mu_\lambda}$;
6 Compute some $\phi_i \leftarrow \mathrm{lift}_{\mu_\lambda}(\psi_i)$;
7 $(\phi_0, n_0) \leftarrow (\phi, n_\lambda)$;
8 $G_0^{(\sigma)}, \dots, G_n^{(\sigma)} \leftarrow$ `Hensel`$(g, \mu_\lambda, [\phi_0^{n_0}, \dots, \phi_s^{n_s}], \sigma)$;
9 Fact, Val $\leftarrow \{\}, \{\}$;
10 **for** $i = 0, \dots, s$ **do**
11 **if** $n_i = 1$ **then**
12 Add $G_i(\sigma)$ to Fact, add μ_λ and $\mu_{G_i^\sigma} = [\mu_\lambda, \mu_{G_i^\sigma}(G_i^{(\sigma)}) = +\infty]$ to Val
13 $\mathrm{Fact}_i, \mathrm{V}_i \leftarrow$ OM-Factorization$(G_i^{(\sigma)}, \sigma)$;
14 Join Fact and Fact_i, join Val and Val_i
15 **return** Fact, Val

Proposition 4.4 ([28]). *Replacing the recursion of Algo. 2 (Line 13) with a divide and conquer strategy, the cost of the OM-factorization of a square-free polynomial $g \in A[x]$ performed at precision $\sigma > \delta(g)$, assuming $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(g)$, is in $O_\varepsilon(\deg(g)\sigma)$ arithmetic operations over κ up to the cost of the residual univariate factorizations.*

5 VALUATIVE TREES AND CLUSTER PICTURES

5.1 The OM valuative tree of a polynomial

The partial order on $V(K[x])$ defined in Subsection 2.1 provides $V(K[x])$ with the structure of a tree in the sense that for any $\mu \in V(K[x])$, the set $\{\rho \in V(K[x]) / \rho \leq \mu\}$ is totally ordered. It is extended with *finite leaves* of the form w_f (see Subsec 2.1) to form the *valuative tree*. It has been studied in more details in [13, §2.4] and prominently in [2].

We present in Algorithm 3 the computation of the sub-tree of the valuative tree obtained from the result of the OM-factorization of a polynomial. We need the following definition for its computation.

Definition 5.1. *A* rooted directed graph *G with root r satisfies that there is a directed path from r to any other vertex of G. The* covering arborescence *of a rooted directed graph G with root r is a subgraph of G such that it contains all the vertices of G but for any other vertex v, there is exactly one directed walk from r to v. It is a directed acyclic graph.*

Remark 5.2. *In SageMath, we can obtain the covering arborescence of G in Algorithm 3 using the* Hasse diagram *method of the* Poset *class.*

Algorithm 3: OM-ValuativeTree

input : g a monic polynomial in $K[x]$, $\sigma \in \mathbb{Q}$ a precision Ensure g is square-free, $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(g)$, and $\sigma > \delta(g)$

output : The valuative tree of g

1 Fact, Val $\leftarrow$ OM-Factorization(g, σ) ;
2 SetVal $\leftarrow$ the set of valuations in the chain of valuations of the inductive valuation μ for all $\mu \in$ Val ;
3 Compute G as the oriented graph whose vertices are the valuations in SetVal and whose oriented edges are the (μ_1, μ_2) when $\mu_1 > \mu_2$ (according to the partial order on $V(K[x])$) ;
4 Compute ValTree as the *covering arborescence* of G with root μ_0 ;
5 **return** ValTree

5.2 From tree to picture

In this subsection, we present how to compute the cluster picture of a polynomial from its valuative tree. The first idea is that the diskoids corresponding to the inductive valuations (thanks to Subsection 2.3) also correspond to the clusters and the Galois orbits of clusters of Section 3, thanks to Prop. 4.2, when the ramification is not wild (see Ex. 4.3). The first ingredient is the following:

Lemma 5.3. *Let us consider an inductive valuation*

$$v_{n+1} = [v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n, v_{n+1}(\phi_{n+1}) = \lambda_{n+1}],$$

with possibly $\lambda_{n+1} = +\infty$. Then $D(\phi_{n+1}, \lambda_{n+1}) \subsetneq D(\phi_n, \lambda_n)$ and in particular, the roots of ϕ_{n+1} are contained in $D(\phi_n, \lambda_n)$.

Proof. Since $v_n \leq v_{n+1}$ then $D(\phi_{n+1}, \lambda_{n+1}) \subsetneq D(\phi_n, \lambda_n)$ by [29, Thm. 4.56]. By Lemma 2.9, the roots of ϕ_{n+1} are contained in $D(\phi_{n+1}, \lambda_{n+1})$ thus in $D(\phi_n, \lambda_n)$. □

Corollary 5.4. *If in the valuative tree of $g \in K[x]$, we have a path $v_0 \to \dots \to v_n \to w_f$ from the root v_0 to some finite leaf w_f for $f \in K^h[x]$ an irreducible factor of g then the roots of f (subset of the roots of g) are contained in all the diskoids of the v_i's.*

Consequently, and thanks to Prop. 4.2, we see that parent-children relations and the branching in the valuative tree expose how the roots of g can be arranged into disks, and more precisely, how one can construct the cluster picture from them. From the previous corollary, the diskoids of v_n contain the roots of f and therefore, we call them the clusters of v_n. They are all conjugated by the Galois action. We continue with a lemma.

Lemma 5.5. *Let us consider an inductive valuation*

$$v_{n+1} = [v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n, v_{n+1}(\phi_{n+1}) = \lambda_{n+1}].$$

obtained as μ on Line 4 of Algorithm 1. Then $\deg(\phi_n)$ divides $\deg(\phi_{n+1})$.

Proof. By the proof of [1, Prop 6.3], $\frac{\deg(\phi_{n+1})}{\deg(\phi_n)} = e_{n+1} f_{n+1} \in \mathbb{Z}$, with e_{n+1}, f_{n+1} the relative ramification index and residual degree of v_{n+1}. □

The integers $\frac{\deg(\phi_{n+1})}{\deg(\phi_n)}$'s will be central in controlling the construction of the clusters along the valuative tree.

To construct the cluster picture of a polynomial $g \in K[x]$ with roots of nonnegative valuation, we start from the root of the valuative tree of g, v_0, the Gauss valuation on the unit disk $D(0,0)$. At first, thanks to Corollary 5.4, all the roots of g are in $D_{\overline{K}}(0,0)$, the diskoid corresponding to v_0. We then follow the descending paths in the valuative tree to refine the diskoids and clusters, arranging the roots inside them.

From a node in the valuative tree, there are two possibilities for its children. If it has one child only, it can be obtained from one step in Algorithm 1. If it has multiple children, they come from dissections in Algorithm 2. In the first case, we can explain how the disks composing the diskoids are refined.

Proposition 5.6. *Let $v_{n+1} = [v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n, v_{n+1}(\phi_{n+1}) = \lambda_{n+1}]$ be an inductive valuation. There are $\frac{\deg(\phi_{n+1})}{\deg(\phi_n)}$ conjugated disks of $D(v_{n+1}, \lambda_{n+1})$ inside each of the disks of $D(v_n, \lambda_n)$.*

Proof. The diskoid D_{v_n} has $\deg(\phi_n)$ disks (in $\overline{K}$) which are all conjugated by Galois action, and same for $D_{v_{n+1}}$ and $\deg(\phi_{n+})$. Because of the Galois action, there are the same number of disks of $D_{v_{n+1}}$ inside each disk of D_{v_n}. Thus there are $\frac{\deg(\phi_{n+1})}{\deg(\phi_n)}$ conjugated disks of $D_{v_{n+1}}$ inside each disk of D_{v_n}. □

Corollary 5.7. *With the same notations, when passing from the node v_n to v_{n+1}, then there are $\frac{\deg(\phi_{n+1})}{\deg(\phi_n)}$ conjugated clusters of v_{n+1} inside each of the clusters of v_n. They may be non-proper clusters.*

We now study what happens in the second case.

Lemma 5.8. *If v_n has $v_{n+1}^{(1)}, \dots, v_{n+1}^{(m)}$ for descendants in the valuative tree, then the diskoids defined by the $v_{n+1}^{(i)}$'s do not intersect.*

Proof. They are disjoints because of [29, Lem. 4.44] and [29, Thm 4.56]. □

Consequently the clusters of v_n are each the home to one cluster of $v_{n+1}^{(j)}$, for all $j \in [1, m]$.

This is enough to get the principle of how to compute a cluster picture from the valuative tree. However, we still need to present a recipe to compute the relative depth of those clusters.

Proposition 5.9. *Let us consider an inductive valuation*

$$v_{n+1} = [v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n, v_{n+1}(\phi_{n+1}) = \lambda_{n+1}].$$

Then the relative depth of the conjugated clusters of v_{n+1} satisfies:

$$\delta_{v_{n+1}} = \lambda_{n+1} - \lambda_n \frac{\deg(\phi_{n+1})}{\deg(\phi_n)}.$$

Proof. Let $\mathfrak{s}_i^{(n)}$ be the roots of ϕ_n inside a cluster of v_i, and same for $\mathfrak{s}_j^{(n+1)}$ with ϕ_{n+1}. Because of Prop 5.6, for all i, $|\mathfrak{s}_i^{(n+1)}| = \frac{\deg(\phi_{n+1})}{\deg(\phi_n)} |\mathfrak{s}_i^{(n)}|$. Also, since ϕ_{n+1} is irreducible, $|\mathfrak{s}_{n+1}^{(n+1)}| = 1$.

By construction, $\delta_{\mathfrak{s}_n^{(n+1)}} = \delta_{v_{n+1}}$. Thanks to [13, Prop. 3.10],

$$\lambda_{n+1} = d_{\mathfrak{s}_0^{(n+1)}} |\mathfrak{s}_0^{(n+1)}| + \sum_{i=1}^{n+1} \delta_{\mathfrak{s}_i^{(n+1)}} |\mathfrak{s}_i^{(n+1)}|,$$

$$\lambda_n = d_{\mathfrak{s}_0^{(n)}} |\mathfrak{s}_0^{(n)}| + \sum_{i=1}^{n-1} \delta_{\mathfrak{s}_i^{(n)}} |\mathfrak{s}_i^{(n)}|.$$

Thus,

$$\begin{aligned}
\lambda_{n+1} &= d_{\mathfrak{s}_0^{(n+1)}} |\mathfrak{s}_0^{(n+1)}| + \sum_{i=1}^{n+1} \delta_{\mathfrak{s}_i^{(n+1)}} |\mathfrak{s}_i^{(n+1)}| \\
&= \frac{\deg(\phi_{n+1})}{\deg(\phi_n)} \left(d_{\mathfrak{s}_0^{(n)}} |\mathfrak{s}_0^{(n)}| + \sum_{i=1}^{n-1} \delta_{\mathfrak{s}_i^{(n)}} |\mathfrak{s}_i^{(n)}| \right) + \delta_{\mathfrak{s}_n^{(n+1)}} \\
&= \delta_{\mathfrak{s}_n^{(n+1)}} + \lambda_n \frac{\deg(\phi_{n+1})}{\deg(\phi_n)},
\end{aligned}$$

from which we can deduce the desired formula. □

Algorithm 4: ClusterPicture

input : g a monic polynomial in $A[x]$, $\sigma \in \mathbb{Q}$ a precision
Ensure g is separable, $\mathrm{char}(\kappa) = 0$ or $\mathrm{char}(\kappa) > \deg(g)$ and $\sigma > \delta(g)$

output : The cluster picture of g

```
ValTree ← OM-ValuativeTree(g, σ) ;
μ ← v_0, the root of ValTree ;
return ClusterPrinting(ValTree, μ)
```

Algorithm 5: ClusterPrinting

input : ValTree, a valuative tree and μ, a valuation, one of the nodes of ValTree

output : The cluster picture of of the valuative tree descending from the node μ

```
if μ = v_0 the root of ValTree then δ ← 0, n_conj ← 1 ;
else
    Write μ = [v_0, ..., v_n(φ_n) = λ_n, v_{n+1}(φ_{n+1}) = λ_{n+1}] ;
    δ ← λ_{n+1} − λ_n deg(φ_{n+1})/deg(φ_n) , n_conj ← deg(φ_{n+1})/deg(φ_n) ;
if μ is a leaf of ValTtree then
    Draw n_conj points connected by "−" and Quit ;
Draw a box 𝔅 with index δ ;
for ν a descendant of μ do
    Call ClusterPrinting(ValTree,ν) to draw inside 𝔅
𝔇 ← 𝔅 ;
for i ∈ ⟦1, n_conj − 1⟧ do        // We compute the orbit
    𝔇 ← 𝔇 − −𝔅 ;                  // of the cluster 𝔅
Draw 𝔇
```

Proposition 5.10. *Algorithm 4 compute the cluster picture of g as long as g satisfies the required conditions of the input.*

Proof. It is clear that the computation provides clusters and orbits of clusters inside the cluster picture of g with the right relative depths. We can add furthermore that no cluster is missing. Indeed, if we assume there is a missing cluster, we can define a diskoid from it, and then an inductive valuation, thanks to the correspondence between diskoids and inductive valuations. This inductive valuation would have to take place in the valuative tree ValTree as it is necessary smaller than some of the valuation at the leaves.

However:
• No valuation providing a dissection in the OM algorithm could have been missed since otherwise, we could find a cycle in the valuative tree $V(K[x])$ using two paths to some leaf of ValTree.
• No valuation can be intermediate between two steps of Algorithm 1. Indeed, this is the main point of [1, §6.1]: no additional refinement step can occur thanks to the usage of approximate roots. □

From Proposition 4.4, we can conclude on the cost of computing the cluster polynomial of a polynomial with moderate ramification:

Theorem 5.11. *For a separable monic polynomial $g \in A[x]$, if either* $\mathrm{char}(\kappa) = 0$ *or* $\mathrm{char}(\kappa) > \deg(g)$*, one can compute the cluster picture of g in $O_\epsilon(\deg(g)\delta(g))$ arithmetic operations over κ (up to the cost of the residual univariate factorizations of the OM algorithm).*

6 BERKOVICH SKELETON OF THE ROOTS

Berkovich geometry is a part of non-archimedean geometry that provides a concept of path connectivity in a world usually totally discontinuous. Being built on norms, its connection to the study of valuations is appealing. A first seed was already present in [17] with its trees of disks. In Rüth's PhD thesis [29], the term "diskoid" was coined because it is related to non-archimedean geometry.

6.1 Berkovich analytification and skeleton

We refer to [4, Chap. 6] for a gentle introduction to Berkovich analytification, Berkovich affine space and Berkovich unit disk.[4]

Definition 6.1. *From* val*, we define a norm $|\cdot|_K$ on K by taking a constant*[5] *$c \in \mathbb{R}_{>0}$ and define for any $x \in K$, $|x|_K = c^{-\mathrm{val}(x)}$. We define the norm $|\cdot|_0$ on $K[x]$ by taking for any $f \in K[x]$, $|f|_0 := \sup_{x \in D(0,0)} |f(x)|_K$.*

Definition 6.2. *The Berkovich unit disk, denoted by $D(0,0)^{an}$, is the set of the multiplicative semi-norms of $K[x]$ bounded by $|\cdot|_0$, i.e. the mappings $|\cdot| : K[x] \to \mathbb{R}_{\geq 0}$ such that for any $(f,g) \in K[x]^2$:*

$$|fg| = |f||g|, \quad |f+g| \leq \max(|f|,|g|) \quad \text{and} \quad |f| \leq |f|_0$$

Definition 6.3. *The mappings $f \mapsto |f(a)|_K = \sup_{D(a,+\infty)} |f|_K$ and $f \mapsto \sup_{D(a,r)} |f|_K$ for $a \in D(0,0)$ and $r \geq 0$ are examples of elements of $D(0,0)^{an}$. The first one is called a point of* Type I *and denoted $|\cdot|_{D(a,+\infty)}$ and the second one is called a point of* Type II *if $r \in \mathrm{val}(K)$ and of* Type III *otherwise, and likewise denoted $|\cdot|_{D(a,r)}$.*

There are also points of Type IV in $D(0,0)^{an}$ but they will play no role for us and thus we skip their introduction.

Definition 6.4. *Let $a, b \in D(0,0)$, $a \neq b$, and $r, s \in \mathbb{R}_+ \cup \{+\infty\}$. Let $t = \min(\mathrm{val}(b-a), r, s)$, the biggest valuation value such that $D(a,r) \subset D(a,t) = D(b,t) \supset D(b,s)$. We define the* path *in $D(0,0)^{an}$ between $|\cdot|_{D(a,r)}$ and $|\cdot|_{D(b,s)}$ as the set $\{|\cdot|_{D(a,u)} \text{ for } u \in [t,r]\} \cup \{|\cdot|_{D(b,s)} \text{ for } u \in [t,s]\}$.*

Definition 6.5. *The* convex hull *of a set $S \subset D(0,0)^{an}$ of points of type I, II or III is the set of all points $|\cdot|_{D(a,u)} \in D(0,0)^{an}$ such that there are $\zeta_1, \zeta_2 \in S$ such that $|\cdot|_{D(a,u)}$ is in the path between ζ_1 and ζ_2. If S is finite, the convex hull of S is called the* skeleton *of S.*

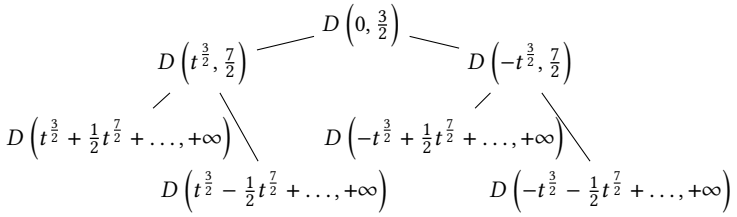
[4]Losing a little of generality in favor of clarity, we have sometimes simplified or weakened the definitions of [4, Chap. 6].
[5]Usually, $c = 2$ or $c = p$ when $K = \mathbb{Q}_p$ for some prime number p.

While not completely unrelated, this definition should not be confused with that of the Berkovich skeleton of a curve or any other notion of skeleton in Berkovich geometry.

6.2 Skeleton of the roots

It is natural to define the *skeleton of the roots* of a polynomial $g \in A[x]$ as the skeleton of the set of its roots in $D_{\overline{K}}(0,0)$ seen as points of type I in $D(0,0)^{an}$.

Example 6.6. *With $f_{1,2} = (x^2 - t^3)^2 - t^{10}$, the skeleton of the roots can be represented as*



From the definition, it is clear that skeleton of the roots can be obtained up to homotopy (*i.e.* without having to represent the roots and the value of the radii) directly from the cluster picture of g by omitting the relative depth and the conjugation between roots, We then get the following complexity result on the cost of the computation of the skeleton of the roots.

Proposition 6.7. *For a separable polynomial $g \in A[x]$, if* $\mathrm{char}(\kappa) = 0$ *or* $\mathrm{char}(\kappa) > \deg(g)$*, one can compute the* skeleton of the set of roots of *g, up to homotopy, with the same cost as in Theorem 5.11.*

This proposition together with Theorem 5.11 leads to the proof of Theorem 1.1. As such, this connection between valuative tree, cluster picture and skeleton of the roots of a polynomial contributes in building a bridge between the theory of valuations and that of Berkovich geometry. For another point of view on this topic, one can also consult [30].

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ANNEX: IMPLEMENTATION

The algorithms presented in this document have been implemented in SageMath [31] and are available here: https://gist.github.com/TristanVaccon.

Basic tools

Most of the low-level components of the algorithms have been implemented by Julian Rüth for SageMath: see https://doc.sagemath.org/html/en/reference/valuations/index.html. In particular, inductive valuations and suitable methods for them were already there, with the exception of the computation of the generalized reduction R_μ for a valuation μ and the truncation of an element of $A[x]$ with respect to a valuation. Liftings and the generalization of Newton polygons were also needed. Here an example of the definition of inductive valuations and an application of R_{v_3}.

```
In:  A = FunctionField(QQ,'t')
In:  B = PolynomialRing(A,'x')
In:  t = A.gen(), x = B.gen()
In:  vA = A.valuation(t)
In:  v0 = GaussValuation(B,vA)
In:  v1 = v0.augmentation(x, 1/2)
In:  v2 = v1.augmentation(x^4 - 2*t^2, 5/2)
In:  v3 = v2.augmentation(x^16 - 8*t^2*x^12 + 24*t^4*x^8
      - 32*t^6*x^4 - 3*t^10 + 16*t^8, 11)
In:  f = ((x^4-2*t^2)^4-3*t^10)^2-6*t^22
In:  Generalized_Reduction(f, v3)
Out: x^2 - 6
```

OM-Irreductibility

We can apply Algorithm 1 to prove the irreducibility over K^h of a polynomial or obtain an inductive valuation proving the reducibility. Here is an example with f_2 defined below.

```
In:  f_2 = ((x^4-2t^2)^4-3t^10)^2-6t^22
In:  test, v, φ = OM_irreducibility_nthRoot_final(f_2)
In:  test
Out: False
In:  v
Out: [ Gauss valuation induced by (t)-adic valuation,
       v(x) = 1/2, v(x^4 - 2 t^2) = 5/2,
       v(x^16 - 8 t^2 x^12 + 24 t^4 x^8 - 32 t^6 x^4 - 3 t^10 + 16 t^8) = 11 ]
```

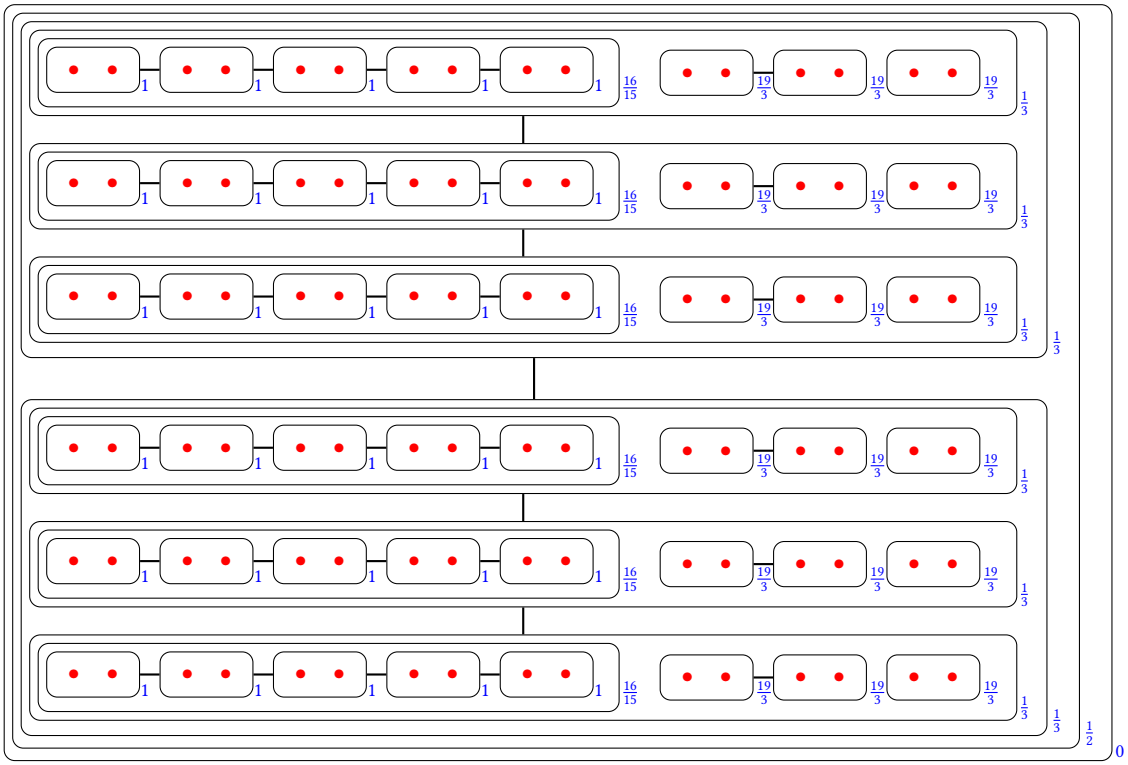


Figure 1: Cluster picture for f_3

OM-Factorisation

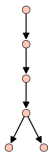
We now showcase an application of Algorithm 2 with the polynomial f_3 which, in order to have non-trivial factors, is defined by: (1) $\phi_1 = x^2 - t$, (2) $\phi_2 = \phi_1^3 + 2\,t^4$, (3) $\phi_3 = (\phi_2^5 + 4\,t^{27})\,(\phi_2^3 - 2\,t^{13}) + t^{51}$, and (4) $f_3 = \phi_3^2 - t^{82} + t^{83}$. Its factorization over the K^h possesses 6 factors. In the following, the factorization takes place with a conservative precision 100.

```
In:  phi1= x^2 - t
In:  phi2 = phi1^3+2*t^4
In:  phi3 = (phi2^5+4*t^27)*(phi2^3-2*t^13) + t^51
In:  f3 = phi3^2-t^82+t^83
In:  irred_fact,G= OM_Factorization(f_3, 100)
In:  [u.degree() for u in irred_fact]
Out: [30, 30, 12, 12, 6, 6]
```

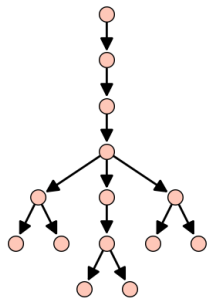
Cluster pictures

We conclude with the display of the valuative trees and cluster pictures of f_2 and f_3. The valuative trees are presented as directed graphs whereas we represent the cluster pictures using ASCII art similarly to [6].

```
In:  VT2 := Valuative_Tree (f2, 50)
Out: Digraph on 6 vertices
In:  VT2.show (vertex_labels = False, layout='tree', tree_root = v0)
```



```
Out:
In:  Cluster_Picture_ASCII_from_Val_Tree(VT)
Out: ((((* *)_1--(* *)_1--(* *)_1--(* *)_1)_1/2
     --((* *)_1--(* *)_1--(* *)_1--(* *)_1)_1/2
     --((* *)_1--(* *)_1--(* *)_1--(* *)_1)_1/2
     --((* *)_1--(* *)_1--(* *)_1--(* *)_1)_1/2)_1/2)_0
In:  VT3 := Valuative_Tree (f3, 100)
Out: Digraph on 14 vertices
In:  VT3.show (vertex_labels = False, layout='tree', tree_root = v0)
```



```
Out:
In:  Cluster_Picture_ASCII_from_Val_Tree(VT3)
Out: ((((((* *)_1--(* *)_1--(* *)_1--(* *)_1--(* *)_1)_16/15
     (* *)_19/3--(* *)_19/3 (* *)_19/3)_1/3--
     (((* *)_1--(* *)_1--(* *)_1--((* *)_19/3
     (* *)_19/3)_1/3)_1/3--((((**)_1--(* *)_1--(* *)_1
     --(* *)_1--(* *)_1)_16/15
      (* *)_19/3--(* *)_19/3 (* *)_19/3)_1/3-_1--(* *)_1)_16/15
     (* *)_19/3--(* *)_19/3 (* *)_19/3)_1/3)_1/3)_1/2)_0
```

A representation of the cluster picture of f_3 is given in Figure 1.